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Construction of a programmed trajectory in the configuration space of rotations for solving the problem of the solid rotation

E.A. Mityushov[1], N.E. Misyura[1], A.E. Lamotkin[1]

[1]The First President of Russia B.N. Yeltsyn Ural Federal University, Russia

Email: mityushov-e@mail.ru, n_misura@mail.ru, alexey.lamotkin@urfu.ru

Abstract. The paper proposes method of programmed control based on the concept of solving the inverse dynamic problem. As a configurational space of rotations, it is proposed to consider a sphere with a radius of 2π in the three-dimensional Euclidean space, which is the image of the unit *Sp(1) quaternions.* A linear relationship has been established between the angular velocity vector of a solid in its spherical motion and the velocity of a point in a sphere allowing to relate the rotation of a solid to the motion of a point inside a three-dimensional sphere. This approach allows to clearly interpret the spherical motion of a solid by the movement of a point inside this sphere, which is used by the authors to describe the rotation of a solid at arbitrary given boundary conditions for angular positions, velocities and accelerations. An example of a smooth turn from one position to another in the case when the turn is set in the sphere in the form of a polynomial of the fifth degree is given.

1. The problem of a smooth rotation of a solid for quaternions

The problem of reorientation of a solid is one of the main problems of the dynamics of a solid. This problem cannot be solved analytically in the general case; possible approaches to solving such problems are considered, for example, in [1-3].

In solving a number of problems associated with the spherical motion of a solid, it was widely used to describe the rotation of a solid in terms of quaternions, which has several advantages over the description in other terms, such as Euler angles.

Kinematic equations expressed in terms of quaternion components are linear

$$\begin{aligned}
\dot{q}_0 &= -\frac{1}{2}\left(q_1\Omega_1 + q_2\Omega_2 + q_3\Omega_3\right),\\
\dot{q}_1 &= \frac{1}{2}\left(q_0\Omega_1 - q_3\Omega_2 + q_2\Omega_3\right),\\
\dot{q}_2 &= \frac{1}{2}\left(q_0\Omega_2 - q_1\Omega_3 + q_3\Omega_1\right),\\
\dot{q}_3 &= \frac{1}{2}\left(q_0\Omega_3 - q_2\Omega_1 + q_1\Omega_2\right),
\end{aligned} \tag{1}$$

where q_0, q_1, q_2, q_3 are the Rodrigue-Hamilton parameters, $\Omega_1, \Omega_2, \Omega_3$ is projections of the angular velocity vector onto the moving coordinate axes here and hereafter.

Consider the problem of reorientation of a solid, the motion of which is determined by equations (1), let the following boundary conditions be imposed on orientation, angular velocity and its derivative:

$$q(0) = q^0, \quad q(T) = q^T, \quad \Omega(0) = \Omega^0, \quad \Omega(T) = \Omega^{\mathrm{T}}, \quad \dot{\Omega}(0) = \dot{\Omega}^0, \quad \dot{\Omega}(T) = \dot{\Omega}^T, \tag{2}$$

where *T is* the maneuver time.
It is required to find a program path and program control minimizing the objective function

$$I_1(q(t), \Omega(t), M(t)), \tag{3}$$

where *M(t) is* the program moment controlling the rotation of the solid.

2. Mapping of the configuration space of rotations to a sphere of radius 2π

Despite the advantages provided by the quaternion parameterization, the task of motion on the hypersphere *Sp(1)* in a four-dimensional space is completely devoid of visibility. These drawbacks can be eliminated by displaying many individual quaternions in three-dimensional space.

The possibility of mapping quaternions defining the orientation of a solid onto a three-dimensional sphere of radius *π* was noted in [4]. In some cases, the configuration space of rotations of a solid can be expanded to a sphere of radius *2π*, in which each unit quaternion

$$q = \cos\frac{\chi}{2} + \sin\frac{\chi}{2}\vec{e},$$

corresponds to a point with a radius vector

$$\vec{r} = \chi\vec{e}.$$

Then, when solving a number of applied problems of motion control, the trajectories in the sphere are continuous and allow you to visually interpret the spherical motion of a solid as the motion of a point along the trajectory in this sphere.

The coordinates of the quaternion are related to the coordinates of the sphere by the following relations

$$\begin{cases} q_0 = \cos\dfrac{(x_1^2 + x_2^2 + x_3^2)^{1/2}}{2}, \\ q_k = \dfrac{x_k(t)}{(x_1^2 + x_2^2 + x_3^2)^{1/2}} \sin\dfrac{(x_1^2 + x_2^2 + x_3^2)^{1/2}}{2}, \quad k = 1,2,3, \end{cases} \tag{4}$$

where

$$x_k = \frac{2q_k \arccos q_0}{(1 - q_0^2)^{1/2}}, \quad k = 1,2,3. \tag{5}$$

3. The problem of the motion of a point inside a three-dimensional sphere

Using the mapping described in the previous paragraph, the optimal rotation problem (1) - (3) can be reformulated as the optimal motion problem for a point in a three-dimensional sphere of radius *2π*.

We write the kinematic relations for determining the projections of the angular velocity on the moving coordinate axes

$$\begin{aligned}\Omega_1 &= 2\left(\dot{q}_1 q_0 - \dot{q}_0 q_1 - \dot{q}_3 q_2 + \dot{q}_2 q_3\right),\\ \Omega_2 &= 2\left(\dot{q}_2 q_0 - \dot{q}_0 q_2 - \dot{q}_1 q_3 + \dot{q}_3 q_1\right),\\ \Omega_3 &= 2\left(\dot{q}_3 q_0 - \dot{q}_0 q_3 - \dot{q}_2 q_1 + \dot{q}_1 q_2\right),\end{aligned} \tag{6}$$

calculating and transforming derivatives (4) we obtain:

$$\dot{q}_0 = -\frac{1}{2}\left(q_1\dot{x}_1 + q_2\dot{x}_2 + q_3\dot{x}_3\right),$$

$$\dot{q}_k = \frac{\dot{x}_k (1-q_0^2)^{1/2}}{2\arccos q_0} + \left(\frac{q_k q_0}{2\left(1-q_0^2\right)} - \frac{q_k}{2\arccos q_0 (1-q_0^2)^{1/2}}\right)\left(q_1\dot{x}_1 + q_2\dot{x}_2 + q_3\dot{x}_3\right), \quad k = 1,2,3.$$

Then, relation (6) can be rewritten in the form of

$$\Omega = A\dot{r}, \tag{7}$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},\ \dot{r} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$

where

$$A_{ii} = \left(\frac{q_0(1-q_0^2)^{1/2}}{\arccos q_0}\right) + \left(\frac{q_i^2 q_0^2}{1-q_0^2}\right) - \left(\frac{q_i^2 q_0}{(1-q_0^2)^{1/2}\arccos q_0}\right) + q_i^2, \qquad i = 1,2,3,$$

$$A_{ij} = \left(\frac{q_k(1-q_0^2)^{1/2}}{\arccos q_0}\right) + \left(\frac{q_i q_j q_0^2}{1-q_0^2}\right) - \left(\frac{q_i q_j q_0}{(1-q_0^2)^{1/2}\arccos q_0}\right) + q_i q_j,$$

here i, j, k is an even permutation of the sequence $1,2,3$,

$$A_{ij} = -\left(\frac{q_k(1-q_0^2)^{1/2}}{\arccos q_0}\right) + \left(\frac{q_i q_j q_0^2}{1-q_0^2}\right) - \left(\frac{q_i q_j q_0}{(1-q_0^2)^{1/2}\arccos q_0}\right) + q_i q_j,$$

here i, j, k is an odd permutation of the sequence $1,2,3$.

Using relation (7), we express the speed of a point in a sphere

$$\dot{r} = A^{-1}\Omega, \tag{8}$$

where

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{22}A_{33} - A_{23}A_{32} & A_{23}A_{31} - A_{21}A_{33} & A_{21}A_{32} - A_{22}A_{31} \\ A_{13}A_{32} - A_{12}A_{33} & A_{11}A_{33} - A_{13}A_{31} & A_{12}A_{31} - A_{11}A_{32} \\ A_{12}A_{23} - A_{13}A_{22} & A_{13}A_{21} - A_{11}A_{23} & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix}.$$

Differentiating and transforming the expression (7) we get:

$$\ddot{r} = A^{-1}\dot{\Omega} - A^{-1}\dot{A}\dot{r}, \tag{9}$$

where

$$\dot{A} = \begin{pmatrix} \dot{A}_{11} & \dot{A}_{12} & \dot{A}_{13} \\ \dot{A}_{21} & \dot{A}_{22} & \dot{A}_{23} \\ \dot{A}_{31} & \dot{A}_{32} & \dot{A}_{33} \end{pmatrix},$$

$$\dot{A}_{ii} = \left(\frac{(\dot{q}_0 (1-q_0^2)^{1/2} + q_0^2 (1-q_0^2)^{-1/2} \dot{q}_0) \arccos q_0 + q_0 \dot{q}_0}{(\arccos q_0)^2} \right) +$$

$$\left(\frac{(2q_i \dot{q}_i q_0^2 + 2q_0 \dot{q}_0 q_i^2)(1-q_0^2) + 4q_i^2 q_0^3 (1-q_0^2) \dot{q}_0}{(1-q_0^2)^2} \right) - \qquad i = 1,2,3,$$

$$\left(\frac{(2q_i \dot{q}_i q_0 + q_i^2 \dot{q}_0)(1-q_0^2)^{1/2} \arccos q_0 - q_i^2 q_0 ((1-q_0^2)^{-1/2} q_0 \dot{q}_0 \arccos q_0 - \dot{q}_0)}{((1-q_0^2)^{1/2} \arccos q_0)^2} \right) + 2q_i \dot{q}_i,$$

$$\dot{A}_{ij} = \left(\frac{(\dot{q}_k (1-q_0^2)^{1/2} + q_k q_0 (1-q_0^2)^{-1/2} \dot{q}_0) \arccos q_0 + q_k \dot{q}_0}{(\arccos q_0)^2} \right) +$$

$$\left(\frac{(\dot{q}_i q_j q_0^2 + q_i \dot{q}_j q_0^2 + 2q_0 \dot{q}_0 q_i q_j)(1-q_0^2) + 4q_i q_j q_0^3 (1-q_0^2) \dot{q}_0}{(1-q_0^2)^2} \right) -$$

$$\left(\frac{(\dot{q}_i q_j q_0 + q_i \dot{q}_j q_0 + q_i q_j \dot{q}_0)(1-q_0^2)^{1/2} \arccos q_0 - q_i q_j q_0 ((1-q_0^2)^{-1/2} q_0 \dot{q}_0 \arccos q_0 - \dot{q}_0)}{((1-q_0^2)^{1/2} \arccos q_0)^2} \right) + \dot{q}_i q_{j+} q_i \dot{q}_j,$$

here i, j, k are an even permutation of the sequence $1,2,3$,

$$\dot{A}_{ij} = -\left(\frac{(\dot{q}_k (1-q_0^2)^{1/2} + q_k q_0 (1-q_0^2)^{-1/2} \dot{q}_0) \arccos q_0 + q_k \dot{q}_0}{(\arccos q_0)^2} \right) +$$

$$\left(\frac{(\dot{q}_i q_j q_0^2 + q_i \dot{q}_j q_0^2 + 2q_0 \dot{q}_0 q_i q_j)(1-q_0^2) + 4q_i q_j q_0^3 (1-q_0^2) \dot{q}_0}{(1-q_0^2)^2} \right) -$$

$$\left(\frac{(\dot{q}_i q_j q_0 + q_i \dot{q}_j q_0 + q_i q_j \dot{q}_0)(1-q_0^2)^{1/2} \arccos q_0 - q_i q_j q_0 ((1-q_0^2)^{-1/2} q_0 \dot{q}_0 \arccos q_0 - \dot{q}_0)}{((1-q_0^2)^{1/2} \arccos q_0)^2} \right) + \dot{q}_i q_{j+} q_i \dot{q}_j,$$

here i, j, k are an odd permutation of the sequence $1,2,3$.

From relation (5), using the boundary conditions (2), we find the boundary positions of the point in the sphere

$$x_k(0) = \frac{2q_k(0) \arccos q_0(0)}{(1-q_0^2(0))^{1/2}}, \quad x_k(T) = \frac{2q_k(T) \arccos q_0(T)}{(1-q_0^2(T))^{1/2}}, \quad k = 1,2,3. \tag{10}$$

Using the boundary conditions (2), matrices $A(0), A(T), A^{-1}(0), A^{-1}(T)$ can be calculated, then using the relation (8) we calculate

$$\dot{r}(0) = A^{-1}(0)\Omega(0), \quad \dot{r}(T) = A^{-1}(T)\Omega(T). \tag{11}$$

Substituting (2) into (1) we find the values $\dot{q}_0(0), \dot{q}_1(0), \dot{q}_2(0), \dot{q}_3(0), \dot{q}_0(T), \dot{q}_1(T), \dot{q}_2(T), \dot{q}_3(T)$ using which we find the matrices $\dot{A}(0)$ and $\dot{A}(T)$. From expression (9) we find

$$\ddot{r}(0) = A^{-1}(0)\dot{\Omega}(0) - A^{-1}(0)\dot{A}(0)\dot{r}(0), \quad \ddot{r}(T) = A^{-1}(T)\dot{\Omega}(T) - A^{-1}(T)\dot{A}(T)\dot{r}(T). \tag{12}$$

Differentiating (5), we obtain

$$\dot{x}_k = \frac{(2\dot{q}_k \arccos q_0 - 2q_k (1-q_0^2)^{-1/2} \dot{q}_0)(1-q_0^2)^{1/2} + 4q_k \arccos q_0 (1-q_0^2)^{-1/2} q_0 \dot{q}_0}{(1-q_0^2)}, \quad k = 1,2,3,$$

After successive substitution of (1) and (4) into this expression, we obtain an equation of the form

$$\dot{x}_k = X_k(x_1, x_2, x_3, \Omega_1, \Omega_2, \Omega_3), \quad k = 1,2,3. \tag{13}$$

Substituting (4) into (3) we obtain the objective function expressed in terms of new parameters

$$I_2(r(t), \Omega(t), M(t)) \tag{14}$$

We reformulate the control problem (1) - (3) in the new parameters. It is required to find the programmed motion of a point in a sphere of radius *2π*, which satisfies equations (13) and boundary conditions (10) - (12), as well as minimizing functional (14).

4. Finding the law of motion of a point inside a three-dimensional sphere in the form of a fifth polynomial

In this paper, to solve the problem, we use a method based on the concept of the inverse dynamics problem and described, for example, in [5,6]. When using this method, the program path is first found that satisfies (10) - (13), after which the control moments are found from the Euler dynamic equations

$$\begin{cases} A\dot{\Omega}_1 + (C-B)\Omega_2\Omega_3 = M_1, \\ B\dot{\Omega}_2 + (A-C)\Omega_1\Omega_3 = M_2, \\ C\dot{\Omega}_3 + (B-A)\Omega_1\Omega_2 = M_3, \end{cases}$$

where M_1, M_2, M_3 are the projections of the control moment M on the axis of the moving coordinate system.

With a sufficient degree of generality, we will assume that the trajectory in the sphere is defined by a fifth-degree vector polynomial

$$r(t) = \sum_{k=0}^{5} a_k t^k, \quad 0 \le t \le T. \tag{15}$$

Their initial conditions (10) - (12), the coefficients of this polynomial are uniquely determined

$$a_0 = r(0),$$
$$a_1 = \dot{r}(0),$$
$$a_2 = \frac{1}{2}\ddot{r}(0),$$
$$a_3 = \frac{10}{T^3}\left(r(T) - r(0) - \dot{r}(0)T - \frac{1}{2}\ddot{r}(0)T^2\right) - \frac{4}{T^2}\left(\dot{r}(T) - \dot{r}(0) - \ddot{r}(0)T\right) + \frac{1}{2T}\left(\ddot{r}(T) - \ddot{r}(0)\right),$$
$$a_4 = -\frac{15}{T^4}\left(r(T) - r(0) - \dot{r}(0)T - \frac{1}{2}\ddot{r}(0)T^2\right) + \frac{7}{T^3}\left(\dot{r}(T) - \dot{r}(0) - \ddot{r}(0)T\right) - \frac{1}{T^2}\left(\ddot{r}(T) - \ddot{r}(0)\right),$$
$$a_5 = \frac{6}{T^5}\left(r(T) - r(0) - \dot{r}(0)T - \frac{1}{2}\ddot{r}(0)T^2\right) - \frac{3}{T^4}\left(\dot{r}(T) - \dot{r}(0) - \ddot{r}(0)T\right) + \frac{1}{2T^3}\left(\ddot{r}(T) - \ddot{r}(0)\right).$$

Thus, there can only be one trajectory defined by a polynomial of the fifth degree and satisfying the boundary conditions. Due to its uniqueness, this trajectory will be optimal regardless of the type of function (14). The main condition for the existence of this trajectory is that the motion (15) must remain inside the sphere with a radius of *2π*, however, this condition can be easily provided with a suitable choice of the parameter *T*.

5. Example

As an example, we consider a smooth (i.e. $\dot{\Omega}^0 = 0, \quad \dot{\Omega}^T = 0$) turn of a solid from orientation $q^0 = (0.5, -0.5, -0.5, -0.5)$ to orientation $q^T = (-0.5, 0.5, 0.5, 0.5)$, with boundary conditions for angular velocities $\Omega^0 = (0.5, 0, 0), \quad \Omega^T = (0, 0, 0.5)$ for a time $T = 10c$.

After calculating the coefficients of polynomial (15), rounding them to the fifth decimal place, we get

$$x_1(t) = -1.2092 + 0.3682t + 0.00872t^2 - 0.02318t^3 + 0.00374t^4,$$
$$x_2(t) = -1.2092 - 0.2364t + 0.041667t^2 + 0.031157t^3 - 0.00489t^4 + 0.00018t^5,$$
$$x_3(t) = -1.2092 + 0.3682t + 0.00872t^2 + 0.00679t^3 - 0.00104t^4 + 0.00003t^5,$$
graphs of these functions are presented in Figure 1.

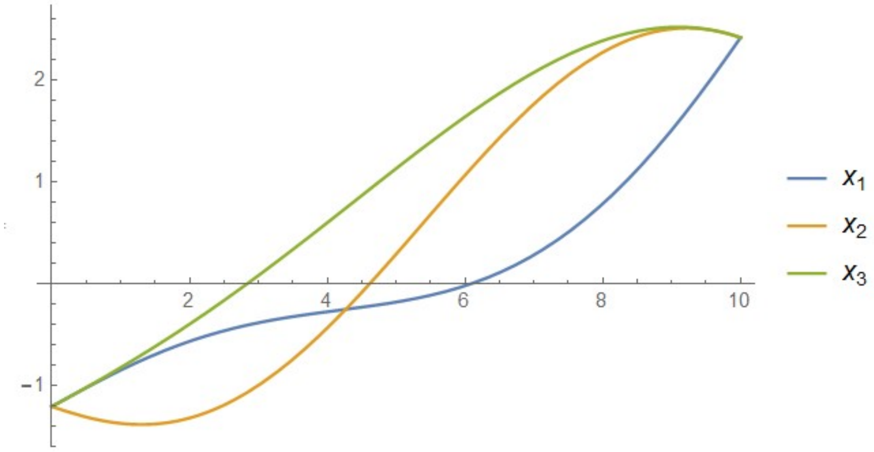

Figure 1. Function charts $x_1(t), x_2(t), x_3(t)$

A visual representation of the trajectory in a sphere of radius *2π* is presented in Figure 3.

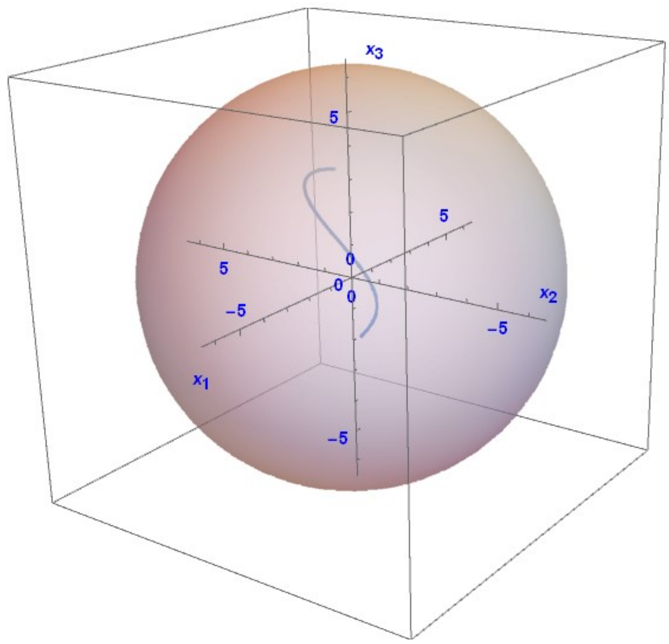

Figure 2. Software path in a sphere with a radius of *2π*

Graphs of the components of the quaternion, angular velocity and its derivative are presented in Figures 3 - 5.

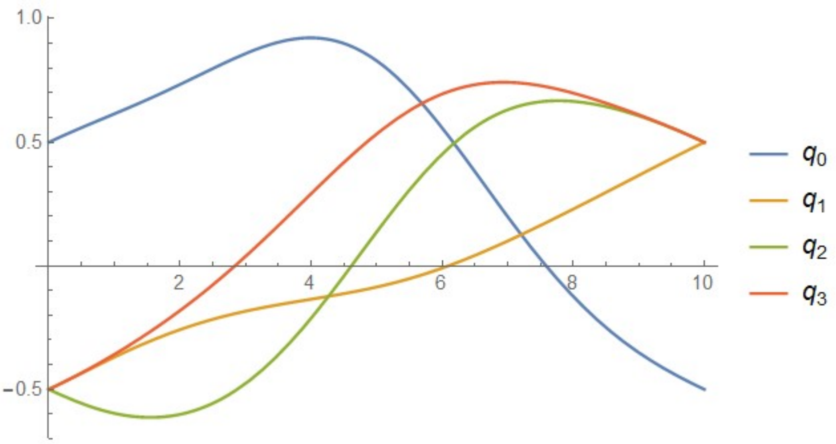

Figure 3. Quaternion component graphs $q_0(t), q_1(t), q_2(t), q_3(t)$

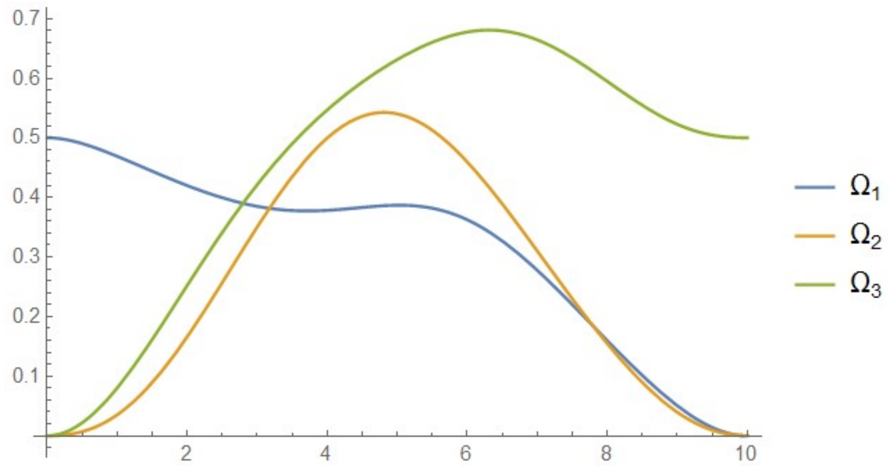

Figure 4. Graphs of angular velocity components $\Omega_1(t), \Omega_2(t), \Omega_3(t)$

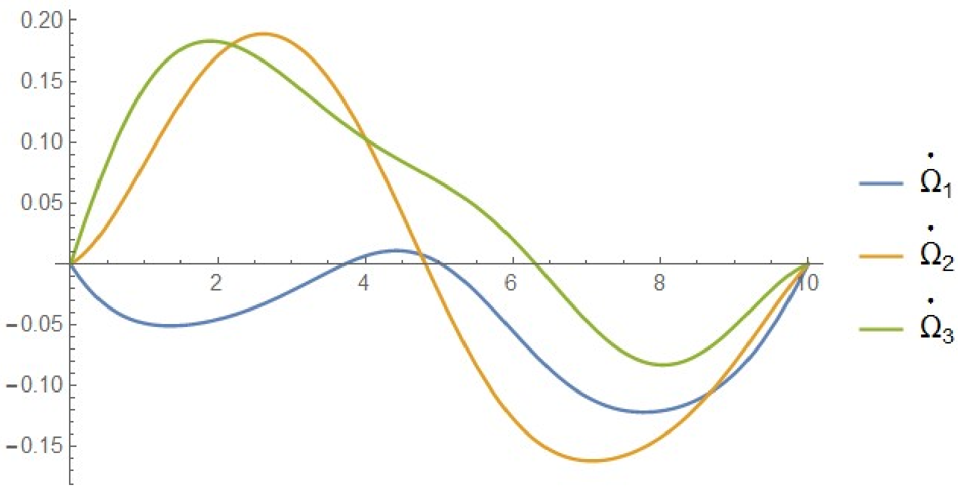

Figure 5. Graphs of the components of the derivative of the angular velocity $\dot{\Omega}_1(t), \dot{\Omega}_2(t), \dot{\Omega}_3(t)$

6. Conclusion

In the work, the problem of optimal solid rotation was considered. A sphere with a radius of *2π is considered as a configuration space of rotations.* The solution of the problem was obtained using the concept of the inverse problem of dynamics in the particular case when the program path is defined by a polynomial of the fifth degree. The considered example demonstrated the success of this approach to solving the problem. In the future, this approach can be expanded to a more general form of program paths and various kinds of objective functions.

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